

# The Geometry of Large Causal Diamonds and the No Hair Property of Asymptotically de-Sitter Spacetimes

G. W. Gibbons

D.A.M.T.P.,

Cambridge University,

Wilberforce Road, Cambridge CB3 0WA, U.K. and

The Galileo Galilei Institute for Theoretical Physics

Arcetri, Firenze, Italy

S. N. Solodukhin

Arnold-Sommerfeld-Center for Theoretical Physics,

Department für Physik,

Ludwig-Maximilians Universität,

Theresienstrasse 37, D-80333, München, Germany

February 5, 2008

## Abstract

In a previous paper we obtained formulae for the volume of a causal diamond or Alexandrov open set  $I^+(p) \cap I^-(q)$  whose duration  $\tau(p, q)$  is short compared with the curvature scale. In the present paper we obtain asymptotic formulae valid when the point  $q$  recedes to the future boundary  $\mathcal{I}^+$  of an asymptotically de-Sitter spacetime. The volume (at fixed  $\tau$ ) remains finite in this limit and is given by the universal formula  $V(\tau) = \frac{4}{3}\pi(2 \ln \cosh \frac{\tau}{2} - \tanh^2 \frac{\tau}{2})$  plus corrections (given by a series in  $e^{-tq}$ ) which begin at order  $e^{-4tq}$ . The coefficients of the corrections depend on the geometry of  $\mathcal{I}^+$ . This behaviour is shown to be consistent with the no-hair property of cosmological event horizons and with calculations of de-Sitter quasinormal modes in the literature.

## 1 Introduction

In a recent paper [1] we embarked on a quantitative study of causal diamonds, or Alexandrov open sets, which are beginning to play an increasingly important role in quantum gravity, for example in the approach via casual sets [2], in

discussions of ‘holography’, and also of the probability of various observations in eternal inflation models (see [3] for a recent example and references to earlier work). The calculations in [1] were concerned with small causal diamonds, that is causal diamonds  $I^+(p) \cap I^-(q)$  whose duration  $\tau(p, q)$ <sup>1</sup> is small compared with the ambient curvature scale. The present paper was motivated by inflationary cosmology and the observations showing that the scale factor  $a(t)$  of our present universe is accelerating. Indeed, it is given to a good approximation by assuming that the spatial geometry is flat and setting the

$$\text{jerk} \equiv \frac{a^2}{\dot{a}^3} \frac{d^3 a}{dt^3} = 1 \quad (1)$$

so that

$$a(\tau) = \sinh^{\frac{2}{3}} \left( \sqrt{\frac{3\Lambda}{4}} t \right), \quad (2)$$

where  $\Lambda$  is the cosmological constant. The jerk is a dimensionless measure of the rate of change of acceleration. It is easily seen to be unity if and only if we have  $k = 0$  model with a cosmological constant and pressure free matter [4, 5, 6, 7]. The physical reason why the jerk of the observed universe is unity is unclear. Equation (2) then solves Einstein’s equations with a cosmological term coupled to a pressure free fluid.

The questions we are interested in concern the observations made by a hypothetical observer moving along a timelike world line  $\gamma$ , in metric which is not exactly but only asymptotically de-Sitter, in the limit that his/her own proper time  $t_q \rightarrow \infty$ . In particular we shall study the volume  $V(\tau, t_q)$  of the causal diamond  $I^+(p) \cap I^-(q)$  where  $p$  and  $q$  lie on  $\gamma$  in the limit when both  $t_p, t_q \rightarrow \infty$  while  $\tau = t_q - t_p$  is kept fixed. Thus both points  $p$  and  $q$  tend to future spacelike infinity  $\mathcal{I}^+$  while the duration of the diamond  $\tau$  is kept fixed. The entire diamond is in the asymptotic region and the volume of the diamond depends on the asymptotic geometry which we wish to explore. The volume (at fixed  $\tau$ ) remains finite in this limit and is given by the universal formula  $V(\tau) = \frac{4}{3}\pi(2 \ln \cosh \frac{\tau}{2} - \tanh^2 \frac{\tau}{2})$  plus corrections (which are given by a series in  $e^{-t_q}$ ) which begin at order  $e^{-4t_q}$ . This behaviour will be shown to be consistent with the no-hair property of cosmological event horizons and with calculations of de-Sitter quasinormal modes in the literature.

Before describing our calculations, we shall give a brief review of the geometry of asymptotically de-Sitter spacetimes.

## 2 Geometry of asymptotically de-Sitter spacetimes

From now on we adopt units in which  $\Lambda = 3$  and hence  $H = 1$ . The metric on de-Sitter space may be cast in Friedmann-Lemaitre form in three different ways

$$(i) \quad k = +1, \quad a(t) = \cosh t, \quad (3)$$

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<sup>1</sup>For relevant definitions and notation the reader is directed to [1].

$$(ii) \quad k = 0, \quad a(t) = \exp t, \quad (4)$$

$$(iii) \quad k = -1, \quad a(t) = \sinh t. \quad (5)$$

Of these only the first is global, that is covers the full geodesically complete spacetime. Another local chart, valid only inside an observer dependent cosmological horizon, is the locally static form

$$ds^2 = -(1 - r^2)dt^2 + \frac{dr^2}{1 - r^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (6)$$

It was conjectured in [9], before the theory of inflation, that perturbations of de-Sitter should settle down inside the the cosmological event horizon ( $r < 1$ ) to the exact static form. How this ‘No Hair’ mechanism works in practice was later elucidated in the context of inflation in [10] who pointed out that while scalar and gravitational perturbations of de-Sitter spacetime described using any of the Friedmann-Lemaître coordinates do not decay, but rather freeze in to constant values at late times, restricted to interior the event horizon of any given inertial observer, the perturbations decay exponentially. One way of understanding this is to note [11] that the general asymptotic form of the metric at late times expressed in quasi-Friedmann-Lemaître, geodesic or Gaussian coordinates takes the form

$$ds^2 = -dt^2 + e^{2t} g_{ij}(x) dx^i dx^j + \dots \quad (7)$$

where  $g_{ij}$  is an *arbitrary* three-metric.

Thus *globally* the metric does not settle down to the de-Sitter form. However *locally* that is within the event horizon of any given observer it does. That is because as time goes on, such an observer can access an exponentially smaller and smaller proportion of the spatial hypersurface  $\Sigma : t = \text{constant}$ . Now provided that the  $\Sigma$  is smooth, no matter what metric it is given, any local patch when examined with sufficient magnification will appear flat. Exact solutions of the Einstein equations describing this process are rather rare, but there are some: the Biaxial Taub-NUT metrics, and that exhibits this mechanism rather clearly [12]. For a recent astrophysical perspective on the eschatology of an asymptotically de-Sitter universes see [18].

In a later, and completely independent, development Fefferman and Graham [13] examined asymptotically hyperbolic Riemannian (i.e positive definite) Einstein metrics with negative scalar curvature near their conformal boundary. It is clear that the asymptotic expansions they obtained are identical in structure to those discussed by Starobinsky earlier [11] for a Lorentzian Einstein metrics with positive scalar curvature near its spacelike conformal boundary. They are also identical in structure to the asymptotical anti-de-Sitter metrics near their timelike boundary [15]. In what follows we shall make use of these expansions. For more work on the de-Sitter case see [16].

We consider a  $(d+1)$ -dimensional space-time which solves the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = \Lambda G_{\mu\nu} \quad (8)$$

with negative cosmological constant  $\Lambda = \frac{d(d-1)}{2l^2}$ ,  $l$  is the de-Sitter radius. We look at the solution to these equations close to the spacelike infinity  $\mathcal{I}^+$  in the form

$$ds^2 = -l^2 \frac{d\rho^2}{2\rho^2} + \frac{1}{\rho} g_{ij}(x, \rho) dx^i dx^j \quad (9)$$

where  $\rho$  is a timelike coordinate such that  $\rho = 0$  at  $\mathcal{I}^+$ . Coordinates  $x^i$ ,  $i = 1, \dots, d$  are the coordinates on the spacelike surface  $\mathcal{I}^+$ . Inserting this metric into the Einstein equations one obtains a system of equations

$$\begin{aligned} \rho [2g'' - 2g'g^{-1}g' + \text{Tr}(g^{-1}g')g'] + l^2 \text{Ric}(g) - (d-2)g' - \text{Tr}(g^{-1}g')g &= 0 \\ \nabla_i \text{Tr}(g^{-1}g') - \nabla^j g'_{ij} &= 0 \\ \text{Tr}(g^{-1}g'') - \frac{1}{2} \text{Tr}(g^{-1}g'g^{-1}g') &= 0, \end{aligned} \quad (10)$$

where differentiation with respect to  $\rho$  is denoted with a prime,  $\nabla_i$  is the covariant derivative constructed from the metric  $g$ , and  $\text{Ric}(g)$  is the Ricci tensor of  $g$ .

Notice that we could have considered the Einstein equations with negative cosmological constant  $\Lambda = -d(d-1)/2l^2$ . The analytic continuation between two cases is a simple replacement  $l^2 \rightarrow -l^2$  both in the metric (9) and in equations (10). The analytic continuation between two spacetimes was considered in detail in the appendix of [19]. Coordinate  $\rho$  then becomes a radial coordinate,  $\rho = 0$  is the timelike infinity of the asymptotically anti-de-Sitter space-time. The solution of the equations (10) in this case is well known in the form of the asymptotic expansion

$$g_{ij}(x, \rho) = g_{ij}^{(0)}(x) + g_{ij}^{(2)}(x)\rho + \dots + g_{ij}^{(d)}(x)\rho^{d/2} + h_{ij}^{(d)}(x)\rho^{d/2} \ln \rho + \dots \quad (11)$$

where  $g_{ij}^{(0)}(x)$  is the metric on the timelike boundary of the anti-de-Sitter space-time. The coefficients  $g_{ij}^{(k)}(x)$ ,  $k < d$  and  $h_{ij}^{(d)}(x)$  are uniquely determined by the metric  $g_{ij}^{(0)}(x)$  while for  $g_{ij}^{(d)}$  only the trace and the covariant divergence are determined by  $g_{ij}^{(0)}(x)$ .  $g_{ij}^{(d)}$  thus encodes the stress energy tensor of the boundary dual theory. Coefficient  $h_{ij}^{(d)}(x)$  is non-vanishing only if  $d$  is even, it has some interesting conformal properties and mathematicians call it the obstruction tensor.

In the asymptotically de-Sitter case one can use same expansion (11) taking into account that  $\rho$  is now a time-like coordinate and metric  $g_{ij}^{(0)}(x)$  is now the metric on the spacelike future infinity  $\mathcal{I}^+$ . Moreover, all expressions for the coefficients  $g_{ij}^{(k)}(x)$  and  $h_{ij}^{(d)}(x)$  as determined by  $g_{ij}^{(0)}(x)$  take exactly same form as in the asymptotically anti-de-Sitter case provided the substitution  $l^2 \rightarrow -l^2$  is applied. In particular we find for the first few coefficients<sup>2</sup> [14], [15],

$$g_{ij}^{(2)}(x) = \frac{l^2}{(d-2)} \left( R_{ij} - \frac{1}{2(d-1)} R g_{ij}^{(0)} \right),$$

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<sup>2</sup>Notice that our curvature notations differ by a sign from those used in [14] and [15].

$$\begin{aligned}
g_{ij}^{(4)} = & \frac{l^4}{(d-4)} \left( \frac{1}{8(d-1)} \nabla_i \nabla_j R - \frac{1}{4(d-2)} \nabla_k \nabla^k R_{ij} \right. \\
& + \frac{1}{8(d-1)(d-2)} \nabla_k \nabla^k R g_{ij}^{(0)} - \frac{1}{2(d-2)} R^{kl} R_{ikjl} \\
& + \frac{d-4}{2(d-2)^2} R_i^k R_{kj} + \frac{1}{(d-1)(d-2)^2} R R_{ij} \\
& \left. + \frac{1}{4(d-2)^2} R^{kl} R_{kl} g_{ij}^{(0)} - \frac{3d}{16(d-1)^2(d-2)^2} R^2 g_{ij}^{(0)} \right) \quad (12)
\end{aligned}$$

in the asymptotically de-Sitter case. The expressions for  $g_{ij}^{(k)}$  are singular when  $k = d$ . In this case  $g_{ij}^{(d)}(x)$  is not uniquely determined by metric  $g_{ij}^{(0)}$ . The Einstein equations impose certain constraints on the trace and covariant divergence of coefficient  $g_{ij}^{(d)}(x)$ .

### 3 Volume of the causal diamond

In the rest of the paper we will be interested in a four-dimensional asymptotically de-Sitter space-time so that  $d = 3$ . Since  $d$  is odd no obstruction tensor appears in the expansion (11). From now on we use units in which  $l = 1$ .

*Asymptotic metric.* In addition to  $\rho$ , two other timelike coordinates can be used. The coordinate  $t$  is defined by relation  $\frac{d\rho^2}{4\rho^2} = dt^2$  so that one has that  $\rho = e^{-2t}$ ,  $t \rightarrow \infty$  at future infinity  $\mathcal{I}^+$ . The coordinate  $t$  is convenient for measuring the geodesic distance (the proper time) along a timelike geodesic. The other coordinate is  $\eta = e^{-t}$ ,  $\eta \geq 0$ ,  $\eta = 0$  at future infinity<sup>3</sup>. In terms of the coordinate  $\eta$  the metric takes the form

$$\begin{aligned}
ds^2 = & \frac{1}{\eta^2} (-d\eta^2 + g_{ij}(x, \eta) dx^i dx^j) \\
g(x, \eta) = & g^{(0)}(x) + g^{(2)}(x) \eta^2 + g^{(3)}(x) \eta^3 + .. \quad (13)
\end{aligned}$$

where one has, as was first shown by Starobinsky [11], that

$$g_{ij}^{(2)}(x) = R_{ij} - \frac{1}{4} R g_{ij}^{(0)}, \quad \text{Tr } g^{(3)} = 0, \quad \nabla^j g_{ij}^{(3)} = 0, \quad (14)$$

where the trace and covariant derivative are defined with respect to metric  $g_{ij}^{(0)}(x)$ . Thus, starting with  $\eta^3$  there appear both even and odd powers of  $\eta$ .

*The Riemann coordinates.* Our coordinate system  $\{x^i\}$  on  $\mathcal{I}^+$  should be adopted to a concrete observer that follows a geodesic  $\gamma$  parameterized by coordinate  $t$ . Suppose that  $\gamma$  intersect  $\mathcal{I}^+$  at a point  $\mathcal{O}$  with coordinates  $x^i = 0$ ,  $i = 1, \dots, d$ .

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<sup>3</sup>Note that we are using a convention in which  $\eta$  is positive and decreases towards future timelike infinity  $\mathcal{I}^+$ .

In a small vicinity of this point one can choose the Riemann coordinate system (for a nice introduction to this coordinate system see [17]) such that

$$\begin{aligned} g_{ij}^{(0)}(x) &= \delta_{ij} - \frac{1}{3}R_{ikjn}(0)x^kx^n - \frac{1}{6}\nabla_k R_{ijnl}(0)x^kx^nx^l + \dots \\ g_{ij}^{(2)}(x) &= (R_{ij}(0) - \frac{1}{4}R(0)\delta_{ij}) + \nabla_k(R_{ij} - \frac{1}{4}R\delta_{ij})(0)x^k + \dots \end{aligned} \quad (15)$$

In terms of the spherical coordinates  $(r, \theta, \phi)$  with centre at  $x = 0$ , one has  $x^k = rn^k(\theta, \phi)$ ,  $k = 1, 2, 3$ , where  $n^k$  is unit vector,  $n^kn^k = 1$ .

*The causal diamond.* We choose the point  $q$  to have coordinates  $(\eta = \epsilon, 0, 0, 0)$  and point  $p$  to have coordinates  $(\eta = N + \epsilon, 0, 0, 0)$ . In terms of coordinate  $t$  we have that  $t_\epsilon = \ln \frac{1}{\epsilon}$  and  $t_{N+\epsilon} = \ln \frac{1}{N+\epsilon}$  so that the proper time interval is  $\tau = t_\epsilon - t_{N+\epsilon} = \ln(\frac{N+\epsilon}{\epsilon})$ . Notice that  $\tau$  can be any finite number. In terms of  $\tau$  one has that  $N = \epsilon(e^\tau - 1)$ . To leading order the equation for the light-cone  $\dot{I}^-(q)$  is

$$r = \eta - \epsilon \quad , \quad 0 \leq r \leq \frac{N}{2}$$

while the equation for the light-cone  $\dot{I}^+(p)$  is

$$r = N + \epsilon - \eta \quad , \quad 0 \leq r \leq \frac{N}{2} \quad .$$

In our calculation we will need the next to leading order modification of the light-cone. In metric (13) the null-geodesic satisfies equation

$$\frac{d\eta}{d\lambda} = \pm \sqrt{g_{ij}(x, \eta)n^in^j} \frac{dr}{d\lambda} \quad , \quad (16)$$

where  $\lambda$  is an affine parameter along the geodesic. To second order in  $r$  and  $\eta$  one has that

$$g_{ij}(x, \eta) = \delta_{ij} - r^2 \frac{1}{3}R_{ikjl}n^kn^l + \eta^2(R_{ij} - \frac{1}{4}R\delta_{ij}) \quad (17)$$

so that

$$g_{ij}n^in^j = 1 + \eta^2(R_{ij}n^in^j - \frac{1}{4}R) \quad .$$

Substituting this into equation (16) and integrating we find the equation for  $\dot{I}^-(q)$ , the past light-cone of  $q$ , up to cubic order in  $\eta$ ,

$$r = r_+(\epsilon) \equiv (\eta - \epsilon) - \frac{1}{6}(R_{ij}n^in^j - \frac{1}{4}R)(\eta^3 - \epsilon^3) \quad , \quad (18)$$

where we took into account the condition that  $r = 0$  when  $\eta = \epsilon$ . A similar equation holds for  $\dot{I}^+(p)$ ,

$$r = r_-(\epsilon) \equiv (N + \epsilon - \eta) + \frac{1}{6}(R_{ij}n^in^j - \frac{1}{4}R)(\eta^3 - (N + \epsilon)^3) \quad . \quad (19)$$

The intersection of the two light-cones,  $\dot{I}^+(p) \cap \dot{I}^-(q)$ , is given by equation

$$\eta = \frac{N}{2} + \epsilon - \frac{1}{8}(R_{ij}n^i n^j - \frac{1}{4}R)N^2(\frac{N}{2} + \epsilon). \quad (20)$$

The correction to the flat space-time result is of order  $\epsilon^3$  and will be neglected in the calculation below.

*The volume.* Consider first the volume of the causal diamond not taking into account the modification of the light-cone. The volume inside the causal diamond is given by expression

$$V_1 = \int_{\epsilon}^{\frac{N}{2}+\epsilon} \frac{d\eta}{\eta^4} \int_0^{\eta-\epsilon} dr r^2 \int_{S_2} \sqrt{\det g} + \int_{\frac{N}{2}+\epsilon}^{N+\epsilon} \frac{d\eta}{\eta^4} \int_0^{N+\epsilon-\eta} dr r^2 \int_{S_2} \sqrt{\det g} \quad (21)$$

where we introduced

$$\int_{S_2} = \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi.$$

To the second order one has that

$$\sqrt{\det g} = 1 - \frac{r^2}{6} R_{kl}(0) n^k(\theta, \phi) n^l(\theta, \phi) + \frac{\eta^2}{8} R(0) + \dots \quad (22)$$

One checks by direct calculation that

$$\int_{S_2} n^k n^l = \frac{4}{3} \pi \delta^{kl}. \quad (23)$$

One thus finds for the volume

$$V_1 = \frac{4\pi}{3} J_1(\tau) - \frac{2\pi}{45} R(0) \epsilon^2 J_2(\tau) + \frac{\pi}{6} R(0) \epsilon^2 J_3(\tau). \quad (24)$$

The contribution to the volume due to the modifications (18) and (19) of light-cones  $\dot{I}^+(p)$  and  $\dot{I}^-(q)$  is given by expression

$$V_2 = \int_{S_2} \left( \int_{\epsilon}^{\frac{N}{2}+\epsilon} \frac{d\eta}{\eta^4} \int_{\eta-\epsilon}^{r+(\epsilon)} dr r^2 + \int_{\frac{N}{2}+\epsilon}^{N+\epsilon} \frac{d\eta}{\eta^4} \int_{\eta-\epsilon}^{r-(\epsilon)} dr r^2 \right). \quad (25)$$

Keeping only terms quadratic in  $\epsilon$  and using identity (23) we find that

$$V_2 = -\frac{\pi}{18} R(0) \epsilon^2 J_4(\tau), \quad (26)$$

where we introduced (recall that  $N = \epsilon(e^\tau - 1)$ )

$$\begin{aligned} \int_{\epsilon}^{\frac{N}{2}+\epsilon} d\eta \left( \frac{1}{\eta^4} + \frac{1}{(N+2\epsilon-\eta)^4} \right) (\eta-\epsilon)^3 &= J_1(\tau) \\ \int_{\epsilon}^{\frac{N}{2}+\epsilon} d\eta \left( \frac{1}{\eta^4} + \frac{1}{(N+2\epsilon-\eta)^4} \right) (\eta-\epsilon)^5 &= \epsilon^2 J_2(\tau) \\ \int_{\epsilon}^{\frac{N}{2}+\epsilon} d\eta \left( \frac{1}{\eta^2} + \frac{1}{(N+2\epsilon-\eta)^2} \right) (\eta-\epsilon)^3 &= \epsilon^2 J_3(\tau) \\ \int_{\epsilon}^{\frac{N}{2}+\epsilon} d\eta \left( \frac{(\eta^3 - \epsilon^3)}{\eta^4} - \frac{(N+2\epsilon-\eta)^3 - (N+\epsilon)^3}{(N+2\epsilon-\eta)^4} \right) (\eta-\epsilon)^2 &= \epsilon^2 J_4(\tau). \end{aligned} \quad (27)$$

The integration can be performed explicitly so that one gets the closed form expressions

$$\begin{aligned}
J_1(\tau) &= 2 \ln \cosh \frac{\tau}{2} - \tanh^2\left(\frac{\tau}{2}\right) \\
J_2(\tau) &= \frac{5}{12}(17e^{2\tau} + 38e^\tau + 17) \tanh^2\left(\frac{\tau}{2}\right) + 10(e^{2\tau} + 1) \ln\left(\frac{1+e^{-\tau}}{2}\right) + 10\tau \\
J_3(\tau) &= \frac{9}{4}(e^\tau - 1)^2 + 3(e^{2\tau} + 1) \ln\left(\frac{1+e^{-\tau}}{2}\right) + 3\tau \\
J_4(\tau) &= \frac{1}{12}(13e^{2\tau} + 10e^\tau + 13) \tanh^2\left(\frac{\tau}{2}\right) + (e^{2\tau} + 1) \ln\left(\frac{1+e^{-\tau}}{2}\right) + \tau \quad (28)
\end{aligned}$$

Notice that there is an identity which holds for these functions

$$-\frac{4}{5}J_2(\tau) + 3J_3(\tau) - J_4(\tau) = 0. \quad (29)$$

Recalling that  $\epsilon = e^{-t_q}$  where  $t_q$  is the time coordinate of the point  $q$  and combining all contributions we obtain the volume as expansion in powers of  $e^{-t_q}$ ,

$$\begin{aligned}
V &= V_1 + V_2 = a_0(\tau) + a_2(\tau)e^{-2t_q} + a_4(\tau)e^{-4t_q} + \dots, \\
a_0(\tau) &= \frac{4\pi}{3}J_1(\tau), \quad a_2(\tau) = \left(-\frac{2\pi}{45}J_2(\tau) + \frac{\pi}{6}J_3(\tau) - \frac{\pi}{18}J_4(\tau)\right)R(0), \quad (30)
\end{aligned}$$

where  $R(0)$  is the Ricci scalar of the 3-dimensional surface  $\mathcal{I}^+$  at the point of intersection of the geodesic  $\gamma$  with  $\mathcal{I}^+$ . Notice that expansion in  $e^{-2t_q}$  is also expansion in the curvature (and its derivatives) of  $\mathcal{I}^+$ . Now, it is a surprising fact that due to identity (29) the coefficient  $a_2(\tau)$  vanishes identically,

$$a_2(\tau) \equiv 0. \quad (31)$$

Notice that the possible term in (30) which is cubic in  $e^{-t_q}$  vanishes. This is due to the fact that in the expansion (13) one has that  $\text{Tr } g^{(3)} = 0$  and due to the property

$$\int_{S_2} n^k n^l n^m = 0.$$

The other possible source for a  $e^{-3t_q}$  term is the  $\epsilon^3$  modification in (20). The analysis however shows that this modification shows up in the volume in the form of even powers of  $\epsilon^3$ , i.e. it may first appear in term  $e^{-6t_q}$ .

So that the next non-vanishing term in expansion (30) is  $e^{-4t_q}$ . It would be interesting to see whether all odd powers of  $e^{-t_q}$  vanish in expansion (30) of the volume. We note that the volume has finite limit when  $t_q \rightarrow \infty$  so no regularization is needed. At first sight, this is surprising since taking that the volume of a bulk region is typically divergent when the boundary of the region approaches infinity (spacelike in anti-de-Sitter and timelike in de-Sitter space-time, see [14] and [15]). However in maximally symmetric spacetime like de-Sitter, it is clear that all causal diamonds with the same duration  $\tau$  are



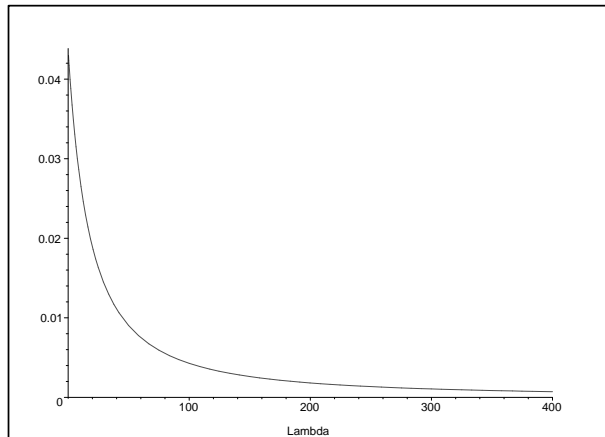


Figure 1: The volume of causal diamond of duration  $\tau = 1$  in pure de-Sitter space-time as function of cosmological constant  $\Lambda$ .

equivalent, no matter how close to future infinity  $\mathcal{I}^+$  they may be, and they must therefore have the same volume  $V(\tau)$  given in fact by the universal formula for  $J_1(\tau)$  in (28). The same universal formula was obtained in a different way in [1] (see (21) of that reference).

We emphasize that the first term in the expansion (30) comes from the metric

$$ds^2 = -dt^2 + e^{2t}(dr^2 + r^2(d\theta^2 + \sin^2 \theta)) \quad (32)$$

of de-Sitter spacetime with flat constant  $t$  slices. This is the only contribution in the limit of  $t_q \rightarrow \infty$ . The curvature of the spacelike surface  $\mathcal{I}^+$  shows up in the  $e^{-2nt_q}$  correction terms. Thus the information on the curvature of  $\mathcal{I}^+$  which is encoded in the volume of the causal diamond is exponentially suppressed. When the diamond as a whole moves closer to the future infinity the geometry inside the diamond becomes more and more accurately de-Sitter. This is of course consistent with results of [10]. There is a nice universality: no matter what is the local geometry in the bulk the geometry inside the diamond becomes de-Sitter when it approaches the future infinity.

At fixed duration  $\tau$  the volume of causal diamond in pure de Sitter space-time becomes a function of the cosmological constant  $\Lambda$ ,

$$V_{\text{ds}}(\tau, \Lambda) = \frac{4\pi}{\Lambda^2} \left( 2 \ln \cosh\left(\frac{\tau\sqrt{3\Lambda}}{6}\right) - \tanh^2\left(\frac{\tau\sqrt{3\Lambda}}{6}\right) \right). \quad (33)$$

In models of eternal inflation  $V(\tau, \Lambda)$  is taken as a measure of probability of an observer of duration  $\tau$ . Thus, we can see how this probability depends on cosmological constant  $\Lambda$ . As is seen in Figure 1 the volume is monotonically decreasing with  $\Lambda$  taking the maximal value at vanishing  $\Lambda$ .

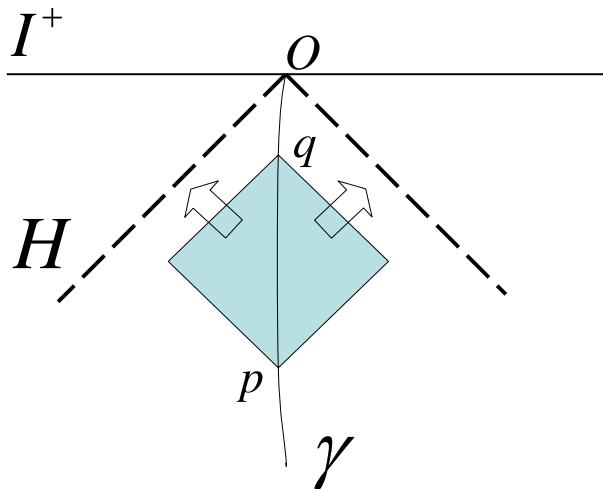


Figure 2: The causal diamond when approaching the future infinity  $\mathcal{I}^+$  becomes more and more accurately described by static de-Sitter coordinates inside the horizon  $\mathcal{H}$  associated with the observer following the timelike geodesic  $\gamma$ . The gravitational perturbations over the de Sitter metric is radiated through the boundary of the diamond. The size of the diamond on the conformal diagram becomes smaller and smaller when  $q$  and  $p$  approach  $\mathcal{O}$ .

## 4 Relation to the quasi-normal modes

There is an alternative way of looking at the time evolution of the geometry inside the causal diamond. In the limit  $t_q, t_p \rightarrow \infty$  the diamond is close to the corner formed by cosmological event horizon  $\mathcal{H}$  of the observer that follows the timelike geodesic  $\gamma$ . Inside this corner one can always take the de-Sitter metric in static coordinate system as a background and consider deviations from the de-Sitter space as perturbations. A perturbation is described by a wave equation and takes the form  $\frac{1}{r} H_l(r) e^{-i\omega t_S} Y_l(\theta, \phi)$ , where  $H_l(r)$  satisfies an effective radial Shrödinger type equation. Inside the diamond these perturbations tend to escape through the boundaries of the diamond. In a bigger picture the perturbations dissipate through the event horizon  $\mathcal{H}$ . The concrete mechanism of the dissipation is given by the quasi-normal modes which are solutions to the gravitational equations for the perturbations subject to condition that they are out-going at the horizon and regular at the origin. This condition can be satisfied only for a discrete complex set of frequencies  $\omega_n$ . For de-Sitter space-time the gravitational quasi-normal modes have been studied for instance in [20] and [21]. An interesting peculiarity of de-Sitter spacetime as compared to a black hole spacetime is that the quasi-normal frequencies are purely imaginary so that they describe the exponential decay only while generically there could be also

oscillations<sup>4</sup>. In  $D$  spacetime dimensions there are two sets of the quasi-normal modes [21]

$$\omega_n = -i(l + D - 1 - q + 2n), \quad \omega_n = -i(l + q + 2n), \quad (34)$$

where  $n = 0, 1, 2, \dots$ ;  $l$  is the angular momentum of the perturbation and the value of  $q$  depends on the type of the perturbation:  $q = 0$  for tensor,  $q = 1$  for vector and  $q = 2$  for scalar perturbations.

The perturbation of the volume of the causal diamond

$$\delta V = \int_{\diamond} \frac{1}{2} \sqrt{G} G^{\mu\nu} H_{\mu\nu} \quad (35)$$

is determined by a scalar type gravitational perturbation. Moreover, since the integration in (35) includes integration over spherical angles then only  $l = 0$  may contribute to the time evolution of the volume.

Let us now compare the two sets (for  $D = 4$ ) of frequencies (34) with our direct calculation (30). Doing this one should keep in mind the relation between global and static coordinate systems. One has that  $\sinh t = (1 - r^2) \sinh t_S$ , where  $t_S$  is time coordinate in static coordinate system and  $t$  is time coordinate in the global coordinate system. Inside the diamond  $r$  changes in the finite limits. Therefore, for large times  $t \sim t_S$ . We see that the set in (34) in which  $i\omega$  is an odd number does not show up in the evolution of the volume. At least this is true for the few lowest frequencies. On the other hand, the second set, in which  $i\omega$  is even number, indeed appears in the evolution of the volume. We have repeated the calculation in the previous section for arbitrary  $D$ . Then the lowest decaying (and, possibly, non-vanishing) term in the volume (30) is of order  $e^{-2t_q}$ . This is again consistent with the second set of quasi-normal frequencies in (34).

## 5 Acknowledgements

This work was initiated at the IHÉS. Both authors would like to thank Thibault Damour and the director Jean Pierre Bourguignon, for their hospitality during our stay at IHÉS. The work was completed while the first author was visiting the Galileo Galilei Institute and he would like to thank the director and the organisers of the the workshop on ‘String and M theory approaches to particle physics and cosmology’ for their hospitality and INFN for partial support. The second author would like to thank Michael Gromov for an interesting discussion.

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<sup>4</sup>The discussion in the literature of the quasi-normal modes in pure de-Sitter spacetime is controversial. This is because there is an apparent cancellation of the would-be quasi-normal poles if  $i\omega$  is an integer and spacetime is pure de-Sitter. On the other hand, as was noted in [20] the presence of an arbitrary small (but non-vanishing) black hole mass prevents  $i\omega$  from being an integer and leads to a negligible correction to the quasi-normal modes.

## References

- [1] G. W. Gibbons and S. N. Solodukhin, The geometry of small causal diamonds, *Phys. Lett. B* in press, [arXiv:hep-th/0703098].
- [2] R. D. Sorkin, Causal sets: Discrete gravity, [arXiv:gr-qc/0309009].
- [3] R. Bousso, R. Harnik, G. D. Kribs and G. Perez, Predicting the cosmological constant from the causal entropic principle, [arXiv:hep-th/0702115].
- [4] T. Chiba and T. Nakamura, “The luminosity distance, the equation of state, and the geometry of the universe,” *Prog. Theor. Phys.* **100**, 1077 (1998) [arXiv:astro-ph/9808022].
- [5] V. Sahni, T. D. Saini, A. A. Starobinsky and U. Alam, “Statefinder – a new geometrical diagnostic of dark energy,” *JETP Lett.* **77**, 201 (2003) [*Pisma Zh. Eksp. Teor. Fiz.* **77**, 249 (2003)] [arXiv:astro-ph/0201498].
- [6] U. Alam, V. Sahni, T. D. Saini and A. A. Starobinsky, “Exploring the Expanding Universe and Dark Energy using the Statefinder Diagnostic,” *Mon. Not. Roy. Astron. Soc.* **344**, 1057 (2003) [arXiv:astro-ph/0303009].
- [7] R. D. Blandford, Measuring and modeling the universe: A theoretical perspective. in W. L. Freedman (Ed.), *Carnegie Observatories Astrophysics Series, Vol. 2: Measuring and Modeling the Universe*, 377-388. Cambridge: Cambridge University Press, 2004.
- [8] G. 't Hooft, Quantum gravity: a fundamental problem and some radical ideas”, in *Recent Developments in Gravitation* (Proceedings of the 1978 Cargèse Summer Institute) edited by M. Levy and S. Deser (Plenum, 1979).
- [9] G. W. Gibbons and S. W. Hawking, Cosmological Event Horizons, Thermodynamics, And Particle Creation,” *Phys. Rev. D* **15** (1977) 2738
- [10] W. Boucher, G. W. Gibbons, Cosmic Baldness, Print-83-0416 (Cambridge U.), June 1982. 6pp. Presented at 1982 Nuffield Workshop on the Very Early Universe, Cambridge, England, Jun 21 - Jul 9, 1982. Published in “The Very Early Universe”, G. W. Gibbons, S. W. Hawking and S. T. C. Siklos (eds), Cambridge University Press (1985).
- [11] A. A. Starobinsky, Isotropization of arbitrary cosmological expansion given an effective cosmological constant, *ZHETF Pis'ma v Redaktsiiu*, **37** (1983) 55-58) translated in *JETP Letters* **37** ( 1983) 66-69. .
- [12] G. W. Gibbons and P. J. Ruback, Classical Gravitons And Their Stability In Higher Dimensions, *Phys. Lett. B* **171** (1986) 390.
- [13] C. Fefferman and C. R. Graham, 1985 Conformal invariants. In: *Elie Cartan et les mathématiques d'aujourd'hui* . Asterisque (hors serie),(1985) 95-116.

- [14] M. Henningson and K. Skenderis, “The holographic Weyl anomaly,” JHEP **9807**, 023 (1998) [arXiv:hep-th/9806087].
- [15] S. de Haro, S. N. Solodukhin and K. Skenderis, Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence, *Commun. Math. Phys.* **217** (2001) 595 [arXiv:hep-th/0002230].
- [16] A. D. Rendall, Asymptotics of solutions of the Einstein equations with positive cosmological constant, *Annales Henri Poincare* **5** (2004) 1041 [arXiv:gr-qc/0312020].
- [17] A. Z. Petrov, ”Einstein spaces” , Pergamon (1969).
- [18] L. M. Krauss and R. J. Sherrer, The Return of a Static Universe and the End of Cosmology [arXiv:0704.02221]
- [19] K. Skenderis, “Lecture notes on holographic renormalization,” Class. Quant. Grav. **19**, 5849 (2002) [arXiv:hep-th/0209067].
- [20] P. R. Brady, C. M. Chambers, W. G. Laarakkers and E. Poisson, “Radiative falloff in Schwarzschild-de-Sitter spacetime,” Phys. Rev. D **60**, 064003 (1999) [arXiv:gr-qc/9902010].
- [21] A. Lopez-Ortega, “Quasinormal modes of D-dimensional de-Sitter spacetime,” Gen. Rel. Grav. **38**, 1565 (2006) [arXiv:gr-qc/0605027]; J. Nataro and R. Schiappa, “On the classification of asymptotic quasinormal frequencies for d-dimensional black holes and quantum gravity,” Adv. Theor. Math. Phys. **8**, 1001 (2004) [arXiv:hep-th/0411267].